

Multiple positive solutions for nonlinear critical fractional elliptic equations involving sign-changing weight functions

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Abstract

In this article, we prove the existence and multiplicity of positive solutions for the following fractional elliptic equation with sign-changing weight functions:

$$\begin{cases} (-\Delta)^\alpha u = a_\lambda(x)|u|^{q-2}u + b(x)|u|^{2_\alpha^*-1}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < \alpha < 1$, Ω is a bounded domain with smooth boundary in \mathbb{R}^N with $N > 2\alpha$ and $2_\alpha^* = 2N/(N - 2\alpha)$ is the fractional critical Sobolev exponent. Our multiplicity results are based on studying the decomposition of the Nehari manifold and the Ljusternik-Schnirelmann category.

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1 Introduction

The fractional Laplacian has attracted much attention recently. It has applications in mathematical physics, biological modeling and mathematical finances and so on. Especially, it appears in turbulence and water wave, anomalous dynamics, flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics, and American options in finance. For more details and applications, see [1, 2, 10, 17, 27, 28] and references therein.

In this paper we focus our attention on critical fractional elliptic problems involving sign-changing functions. More precise, we consider the following elliptic equation involving the fractional Laplacian:

$$\begin{cases} (-\Delta)^\alpha u = a_\lambda(x)|u|^{q-2}u + b(x)|u|^{2_\alpha^*-1}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain with smooth boundary in \mathbb{R}^N , $0 < \alpha < 1$, $N > 2\alpha$, $1 < q < \min\{2, 2_\alpha^* - 1\}$, $\lambda > 0$ is real parameter and $2_\alpha^* = \frac{2N}{N-2\alpha}$ is the fractional critical Sobolev exponent. Here $(-\Delta)^\alpha$ is the fractional Laplacian defined, up to a normalization constant, as

$$(-\Delta)^\alpha u(x) = P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} dy, \quad (1.2)$$

for $x \in \mathbb{R}^N$, where P.V. denotes the principal value of the integral.

Concerning the weight functions $a_\lambda(x)$ and $b(x)$, we may assume that

(H_1) $a_\lambda = \lambda a_+ + a_-$, with $a_\pm = \pm \max\{\pm a, 0\} \not\equiv 0$, and b are continuous in $\bar{\Omega}$;

(H_2) There exists a nonempty closed set

$$M = \left\{ x \in \bar{\Omega} \mid b(x) = \max_{\bar{\Omega}} b \equiv 1 \right\} \subset \Omega$$

and a positive number $t > \frac{N-2\alpha}{2}$ such that

$$b(z) - b(x) = O(|x - z|^t)$$

holds uniformly for $z \in M$ in the limit $x \rightarrow z$.

Remark 1.1 *Let*

$$M_r = \{x \in \mathbb{R}^N \mid \text{dist}(x, M) < r\} \quad \text{for } r > 0.$$

By (H_2), we may then assume that there exist constants η_0, D_0 and r_0 such that

$$b(x) \geq \eta_0 \quad \text{for all } x \in M_{r_0} \subset \Omega.$$

and

$$b(z) - b(x) \leq D_0 |x - z|^t \quad \text{for all } x \in B_{r_0}(z) \text{ and for all } z \in M.$$

When $a_\lambda \equiv \lambda$ and $b \equiv 1$, problem (1.1) has been studied by Barrios et al. in [4]. They proved that there exists a positive Λ such that (1.1) admits at least two solutions if $\lambda \in (0, \Lambda)$. One can also define a fractional power of the Laplacian using spectral decomposition. Problem (1.1) for the spectral fractional Laplacian has been treated in [5]. In this article, we study problem (1.1) with sign-changing weight functions. Our first main result is

Theorem 1.1 *Suppose that (H_1) and (H_2) hold. Let*

$$\Lambda_0 = \frac{q}{2} \cdot \frac{S_\alpha^{\frac{N(2-q)}{4\alpha} + \frac{q}{2}}}{\|a_+\|_{L^{q^*}(\Omega)}} \cdot \left(\frac{2-q}{2_\alpha^* - q} \right)^{\frac{(N-2\alpha)(2-q)}{4\alpha}} \cdot \left(\frac{2_\alpha^* - 2}{2_\alpha^* - q} \right),$$

where S_α is the best Sobolev constant for the embedding of $H^\alpha(\mathbb{R}^N)$ into $L^{2_\alpha^*}(\mathbb{R}^N)$ (see (2.1) below) and $q^* = 2_\alpha^*/(2_\alpha^* - q)$. Then problem (1.1) has at least two positive solutions if $\lambda \in (0, \Lambda_0)$.

We use variational methods to find positive solutions of equation (1.1). We denote by $H^\alpha(\mathbb{R}^N)$ the usual fractional Sobolev space endowed with the so-called Gagliardo norm

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy \right)^{1/2}. \quad (1.3)$$

while $X_0^\alpha(\Omega)$ is the function space defined as

$$X_0^\alpha(\Omega) = \{u \in H^\alpha(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}. \quad (1.4)$$

In $X_0^\alpha(\Omega)$ we consider the following norm

$$\|u\|_{X_0^\alpha(\Omega)} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy \right)^{1/2}. \quad (1.5)$$

We also recall that $(X_0^\alpha(\Omega), \|\cdot\|_{X_0^\alpha(\Omega)})$ is a Hilbert space with scalar product

$$\langle u, v \rangle_{X_0^\alpha(\Omega)} = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2\alpha}} dx dy, \quad (1.6)$$

see Lemma 7 in [23]. Note that by Proposition 3.6 in [13] we have the following identities, up to constants,

$$\|u\|_{X_0^\alpha(\Omega)} = \| |\xi|^\alpha \mathcal{F}u \|_{L^2(\mathbb{R}^N)} = \| \mathcal{F}(-\Delta)^{\alpha/2} u \|_{L^2(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 dx \right)^{1/2}.$$

We have used that if u and v in $X_0^\alpha(\Omega)$, then

$$\int_{\Omega} v(-\Delta)^\alpha u dx = \int_{\mathbb{R}^N} (-\Delta)^{\alpha/2} v (-\Delta)^{\alpha/2} u dx$$

which yields the following definition

Definition 1.1 We say that $u \in X_0^\alpha(\Omega)$ is a weak solution of (1.1) if for every $\varphi \in X_0^\alpha(\Omega)$, one has

$$\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2\alpha}} dx dy = \int_{\Omega} a_\lambda |u|^{q-1} u \varphi dx + \int_{\Omega} b |u|^{p-1} u \varphi dx. \quad (1.7)$$

In this sequel we will omit the term weak when referring to solutions that satisfy the conditions of Definition 1.1. Associated with equation (1.1), we consider the energy functional Φ_λ in $X_0^\alpha(\Omega)$,

$$\Phi_\lambda(u) = \frac{1}{2} \|u\|_{X_0^\alpha(\Omega)}^2 - \frac{1}{q} \int_{\Omega} a_\lambda |u|^q dx - \frac{1}{2_\alpha^*} \int_{\Omega} b |u|^{2_\alpha^*} dx.$$

As it is well known, when one uses the variational methods to find the critical points of the functional, some geometry structures are needed such as the mountain pass structure, the linking structures and so on. For problem (1.1), the main difficulty lies in the functional may not posses such structures since the sign-changing weight. In order to overcome this difficulty, we turn to another approach, that is, the Nehari manifold, which was introduced by Nehari in [18] and has been widely used in the literature, for example [26, 3, 30, 31, 32] and references therein for Laplace operator and also [7, 33] for the fractional Laplacian. The main idea of these articles lies in dividing the Nehari manifold into three parts and considering the infima of the functional on each part. More precise, the Nehari manifold for $\Phi_\lambda(u)$ is defined as

$$\begin{aligned} \mathcal{N}_\lambda &= \{u \in X_0^\alpha(\Omega) \mid \langle \Phi'_\lambda(u), u \rangle = 0\} \\ &= \left\{ u \in X_0^\alpha(\Omega) \mid \|u\|_{X_0^\alpha(\Omega)}^2 - \int_{\Omega} a_\lambda |u|^q dx - \int_{\Omega} b |u|^{2_\alpha^*} dx = 0 \right\}. \end{aligned}$$

It is clear that all critical points of Φ must be lie on \mathcal{N}_λ , as we will see below, local minimizers on \mathcal{N}_λ are usually critical points of Φ_λ . By consider the fibering map $h_u(t) = \Phi_\lambda(tu)$, we can divide that \mathcal{N}_λ into three subsets $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^-$ and \mathcal{N}_λ^0 which correspond to local minima, local maxima and points of inflexion of fibbering maps. Then we can find that $\mathcal{N}_\lambda^0 = \emptyset$ if $\lambda \in (0, \Lambda_0)$ and meanwhile there exists at least one positive solution in \mathcal{N}_λ^+ and \mathcal{N}_λ^- respectively. Moreover, by applying the Ljusternik-Schnirelmann category (see for example [15]), we can show another multiplicity result. We would like point out that, if Y is a closed subset of a topological space X , the Lusternik-Schnirelman category $cat_X(Y)$ is the least number of closed and contractible sets in X which cover Y . Here and in what follows, we denote cat as the Ljusternik-Schnirelmann category. Recalling the definition of M and M_δ in (H_2) and Remark 1.1 respectively and using the Ljusternik-Schnirelmann category, we can prove that

Theorem 1.2 *Suppose that (H_1) and (H_2) hold. For each $\delta < r_0$ (see Remark 1.1), then there exists $0 < \Lambda_\delta \leq \Lambda_0$ such that problem (1.1) has at least $\text{cat}_{M_\delta}(M) + 1$ positive solutions for each $\lambda \in (0, \Lambda_\delta)$.*

When $\alpha = 1$ and $u = 0$ on $\partial\Omega$, de Pavia [19] studied sufficient small λ and obtained a globalized result, indicating that there exists a λ^* such that (1.1) has at least two solutions if $\lambda \in (0, \lambda^*)$. In [19], they requires that one of the weight functions is non-negative with a non-empty domain for which $a(x)$ and $b(x)$ are both positive. In order to overcome the nonnegative assumptions on the weight functions, Chen et al. [8] recently by studying the decomposition of the Nehari manifold relaxed the conditions of the weight functions set out by de Pavia [19] with hypotheses $(H_1) - (H_2)$ (without imposing the non-negativity constraint on the weight functions $a(x)$ and $b(x)$) and investigate the solution structure of (1.1). This method is also used in [30, 31, 32, 26, 3] and reference therein. Furthermore, in [8] the authors also proved there exists at least $\text{cat}_{M_\delta}(M) + 1$ positive solutions based on the concentration-compactness principle and the Lusternik-Schnirelman category. The concentration-compactness principle for the fractional Laplacian is obtained by Palatucci and Pisante [20] recently. Thus, we would like to extend the result in [8] to equation (1.1), Theorem 1.2.

This article is organized as follows. In Section 2 we give some notations and preliminaries for the Nehari manifold. Sections 3 and 4 are devoted to prove the multiplicity of positive solutions of equation (1.1), Theorem 1.1 and Theorem 1.2, respectively.

2 Preliminaries

We start this section by recalling the best Sobolev constant S_α for the embedding of $H^\alpha(\mathbb{R}^N)$ into $L^{2^*_\alpha}(\mathbb{R}^N)$, which is defined as

$$S_\alpha = \inf_{H^\alpha(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy}{\left(\int_{\mathbb{R}^N} |u|^{2^*_\alpha} dx \right)^{2/2^*_\alpha}} > 0. \quad (2.1)$$

By Theorem 1.1 in [11], the infimum in (2.1) is attained at the function

$$u_0(x) = \kappa / (|x - x_0|^2 + \mu^2)^{(N-2\alpha)/2} \quad (2.2)$$

where $\kappa \in \mathbb{R}$, $\mu > 0$ and $x_0 \in \mathbb{R}^N$ are fixed constants. Moreover, let

$$\tilde{u}(x) = u_0(x/S_\alpha^{1/2\alpha}) \quad \text{for } x \in \mathbb{R}^N,$$

then \tilde{u} is a positive solution of the critical problem

$$(-\Delta)^\alpha u = |u|^{2_\alpha^*-1} u \quad \text{in } \mathbb{R}^N. \quad (2.3)$$

Furthermore, for any $\varepsilon > 0$, we define

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-2\alpha}{2}} \tilde{u}(x/\varepsilon) \quad \text{for } x \in \mathbb{R}^N, \quad (2.4)$$

then U_ε satisfying (2.3) and also

$$\|U_\varepsilon\|_{H^\alpha(\mathbb{R}^N)}^2 = \|U_\varepsilon\|_{L^{2_\alpha^*}(\mathbb{R}^N)}^{2_\alpha^*} = S_\alpha^{N/2\alpha}.$$

We define the Palais-Smale (PS)-sequences and (PS)-condition in $X_0^\alpha(\Omega)$ for Φ_λ as follows.

Definition 2.1 (1) For $c \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_c$ -sequence in $X_0^\alpha(\Omega)$ for Φ_λ if $\Phi_\lambda(u_n) = c + o(1)$ and $\Phi'_\lambda(u_n) = o(1)$ strongly in $(X_0^\alpha(\Omega))^*$ as $n \rightarrow \infty$.

(2) Φ_λ satisfies the $(PS)_c$ -condition in $X_0^\alpha(\Omega)$ if every $(PS)_c$ -sequence in $X_0^\alpha(\Omega)$ for Φ_λ contains a convergent subsequence.

Since the energy functional Φ_λ is not bounded below on $X_0^\alpha(\Omega)$, it is useful to consider the functional on the Nehari manifold \mathcal{N}_λ . Moreover, we have the following result.

Lemma 2.1 The energy functional Φ_λ is coercive and bounded below on \mathcal{N}_λ .

Proof. By Hölder and Sobolev inequalities, for $u \in \mathcal{N}_\lambda$, we have

$$\Phi_\lambda(u) = \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \|u\|_{X_0^\alpha(\Omega)}^2 - \left(\frac{1}{q} - \frac{1}{2_\alpha^*}\right) \int_\Omega (\lambda a_+ + a_-) |u|^q dx \quad (2.5)$$

$$\begin{aligned} &\geq \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \|u\|_{X_0^\alpha(\Omega)}^2 - \left(\frac{1}{q} - \frac{1}{2_\alpha^*}\right) \int_\Omega \lambda a_+ |u|^q dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \|u\|_{X_0^\alpha(\Omega)}^2 - \lambda \left(\frac{1}{q} - \frac{1}{2_\alpha^*}\right) \|a_+\|_{L^{q^*}(\Omega)} S_\alpha^{-\frac{q}{2}} \|u\|_{X_0^\alpha(\Omega)}^q \end{aligned} \quad (2.6)$$

where $q^* = 2_\alpha^*/(2_\alpha^* - q)$. Then Φ_λ is coercive and bounded below on \mathcal{N}_λ . \square

The Nehari manifold \mathcal{N}_λ is closely related to the behaviour of the function of the form $h_u : t \rightarrow \Phi_\lambda(tu)$ for $t > 0$. Such map are know as fibering maps

that dates back to the fundamental works [21, 9, 22, 12]. If $u \in X_0^\alpha(\Omega)$, we have

$$\begin{aligned} h_u(t) &= \frac{t^2}{2} \|u\|_{X_0^\alpha(\Omega)}^2 - \frac{t^q}{q} \int_{\Omega} a_\lambda |u|^q dx - \frac{t^{2_\alpha^*}}{2_\alpha^*} \int_{\Omega} b |u|^{2_\alpha^*} dx; \\ h'_u(t) &= t \|u\|_{X_0^\alpha(\Omega)}^2 - t^{q-1} \int_{\Omega} a_\lambda |u|^q dx - t^{2_\alpha^*-1} \int_{\Omega} b |u|^{2_\alpha^*} dx; \\ h''_u(t) &= \|u\|_{X_0^\alpha(\Omega)}^2 - (q-1)t^{q-2} \int_{\Omega} a_\lambda |u|^q dx - (2_\alpha^*-1)t^{2_\alpha^*-2} \int_{\Omega} b |u|^{2_\alpha^*} dx. \end{aligned}$$

We observe that

$$h'_u(t) = \langle \Phi'_\lambda(tu), u \rangle = \frac{1}{t} \langle \Phi'_\lambda(tu), tu \rangle$$

and thus, for $u \in X_0^\alpha(\Omega) \setminus \{0\}$ and $t > 0$, $h'_u(t) = 0$ if and only if $tu \in \mathcal{N}_\lambda$, that is, positive critical points of h_u correspond points on the Nehari manifold. In particular, $h'_u(1) = 0$ if and only if $u \in \mathcal{N}_\lambda$. So it is natural to split \mathcal{N}_λ into three parts corresponding local minimal, local maximum and points of inflection. Accordingly, we define

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda \mid h''_u(1) > 0\}; \\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda \mid h''_u(1) = 0\}; \\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda \mid h''_u(1) < 0\}. \end{aligned}$$

Next, we establish some basic properties of \mathcal{N}_λ^+ , \mathcal{N}_λ^0 , and \mathcal{N}_λ^- .

Lemma 2.2 *Suppose that u_0 is a local minimizer of Φ_λ on \mathcal{N}_λ and $u_0 \notin \mathcal{N}_\lambda^0$. Then $\Phi'_\lambda(u_0) = 0$ in $(X_0^\alpha(\Omega))^*$, where $(X_0^\alpha(\Omega))^*$ is the dual space of $X_0^\alpha(\Omega)$.*

Proof. If u_0 is a local minimizer for Φ_λ on \mathcal{N}_λ , then u_0 is a solution of the optimization problem

$$\text{minimizer } \Phi_\lambda(u) \text{ subject to } J(u) = 0,$$

where $J(u) = \|u\|_{X_0^\alpha(\Omega)}^2 - \int_{\Omega} a_\lambda |u|^q dx - \int_{\Omega} b |u|^{2_\alpha^*} dx$. Hence, by the theory of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $\Phi'_\lambda(u_0) = \mu J'(u_0)$. Thus we have

$$\langle \Phi'_\lambda(u_0), u_0 \rangle = \mu \langle J'(u_0), u_0 \rangle. \quad (2.7)$$

Since $u_0 \in \mathcal{N}_\lambda$, we have that $\|u_0\|_{X_0^\alpha(\Omega)}^2 - \int_{\Omega} a_\lambda |u_0|^q dx - \int_{\Omega} b |u_0|^{2_\alpha^*} dx = 0$. Hence,

$$\begin{aligned} \langle J'(u_0), u_0 \rangle &= 2\|u_0\|_{X_0^\alpha(\Omega)}^2 - q \int_{\Omega} a_\lambda |u_0|^q dx - 2_\alpha^* \int_{\Omega} b |u_0|^{2_\alpha^*} dx \\ &= \|u_0\|_{X_0^\alpha(\Omega)}^2 - (q-1) \int_{\Omega} a_\lambda |u_0|^q dx - (2_\alpha^*-1) \int_{\Omega} b |u_0|^{2_\alpha^*} dx. \end{aligned}$$

So, if $u_0 \notin \mathcal{N}_\lambda^0$, $\langle J'(u_0), u_0 \rangle \neq 0$ and thus $\mu = 0$ by (2.7). Hence, we complete the proof. \square

For each $u \in \mathcal{N}_\lambda$, we know that

$$\begin{aligned} h_u''(1) &= \|u\|_{X_0^\alpha(\Omega)}^2 - (q-1) \int_\Omega a_\lambda |u|^q dx - (2_\alpha^* - 1) \int_\Omega b |u|^{2_\alpha^*} dx \\ &= (2 - 2_\alpha^*) \|u\|_{X_0^\alpha(\Omega)}^2 - (q - 2_\alpha^*) \int_\Omega a_\lambda |u|^q dx \end{aligned} \quad (2.8)$$

$$= (2 - q) \|u\|_{X_0^\alpha(\Omega)}^2 - (2_\alpha^* - q) \int_\Omega b |u|^{2_\alpha^*} dx. \quad (2.9)$$

Then we have following result.

Lemma 2.3 (1) For any $u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$, we have $\int_\Omega a_\lambda |u|^q dx > 0$;
(2) For any $u \in \mathcal{N}_\lambda^-$, we have $\int_\Omega b |u|^{2_\alpha^*} dx > 0$.

Proof. By the definitions of \mathcal{N}_λ^+ and \mathcal{N}_λ^0 , it is easy to get that $\int_\Omega a_\lambda |u|^q dx > 0$ from (2.8). Similarly, the definition of \mathcal{N}_λ^- and (2.9) imply that $\int_\Omega b |u|^{2_\alpha^*} dx > 0$. \square

Let $\Lambda_1 = \frac{S_\alpha^{\frac{N(2-q)}{4\alpha} + \frac{q}{2}}}{\|a_+\|_{L^{q^*}(\Omega)}} \cdot \left(\frac{2-q}{2_\alpha^*-q}\right)^{\frac{(N-2\alpha)(2-q)}{4\alpha}} \cdot \left(\frac{2_\alpha^*-2}{2_\alpha^*-q}\right)$. Then we have the following result.

Lemma 2.4 We have $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda < \Lambda_1$.

Proof. We prove it by contradiction arguments. Suppose that there exists $\lambda < \Lambda_1$ such that $\mathcal{N}_\lambda^0 \neq \emptyset$. Then, for $u_0 \in \mathcal{N}_\lambda^0$, by (2.8) and the Hölder and Sobolev inequalities, we have

$$\begin{aligned} \|u\|_{X_0^\alpha(\Omega)}^2 &= \frac{2_\alpha^* - q}{2_\alpha^* - 2} \int_\Omega a_\lambda |u|^q dx \\ &\leq \frac{2_\alpha^* - q}{2_\alpha^* - 2} \int_\Omega \lambda a_+ |u|^q \\ &\leq \lambda \cdot \frac{2_\alpha^* - q}{2_\alpha^* - 2} \|a_+\|_{L^{q^*}(\Omega)} \|u\|_{L^{2_\alpha^*}(\Omega)}^q \\ &\leq \lambda \cdot \frac{2_\alpha^* - q}{2_\alpha^* - 2} \|a_+\|_{L^{q^*}(\Omega)} S_\alpha^{-\frac{q}{2}} \|u\|_{X_0^\alpha(\Omega)}^q \end{aligned}$$

and so

$$\|u\|_{X_0^\alpha(\Omega)}^{2-q} \leq \lambda \cdot \frac{2_\alpha^* - q}{2_\alpha^* - 2} \|a_+\|_{L^{q^*}(\Omega)} S_\alpha^{-\frac{q}{2}}. \quad (2.10)$$

Similarly, by (2.9) the Hölder and Sobolev inequalities, we have

$$\|u\|_{X_0^\alpha(\Omega)} \geq \left(\frac{2-q}{2_\alpha^* - q} \right)^{\frac{(N-2\alpha)}{4\alpha}} S_\alpha^{\frac{N}{4\alpha}} \quad (2.11)$$

since $\max_{\bar{\Omega}} b(x) \equiv 1$.

Hence, combining (2.10) and (2.11), we must have

$$\lambda \geq \frac{S_\alpha^{\frac{N(2-q)}{4\alpha} + \frac{q}{2}}}{\|a_+\|_{L^{q^*}(\Omega)}} \cdot \left(\frac{2-q}{2_\alpha^* - q} \right)^{\frac{(N-2\alpha)(2-q)}{4\alpha}} \cdot \left(\frac{2_\alpha^* - 2}{2_\alpha^* - q} \right) = \Lambda_1,$$

which is a contradiction. This completes the proof. \square

In order to get a better understanding of the Nehari manifold and the fibering maps, we considering the function $m_u : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by

$$m_u(t) = t^{2-q} \|u\|_{X_0^\alpha(\Omega)}^2 - t^{2_\alpha^*-q} \int_{\Omega} b|u|^{2_\alpha^*} dx \quad \text{for } t > 0. \quad (2.12)$$

It is clear that $tu \in \mathcal{N}_\lambda$ if and only $m_u(t) = \int_{\Omega} a_\lambda |u|^q$. Moreover,

$$m'_u(t) = (2-q)t^{1-q} \|u\|_{X_0^\alpha(\Omega)}^2 - (2_\alpha^* - q)t^{2_\alpha^*-q-1} \int_{\Omega} b|u|^{2_\alpha^*} dx \quad (2.13)$$

and it is easy to see that, if $tu \in \mathcal{N}_\lambda$, then $t^{q-1}m'_u(t) = h''_u(t)$. Hence $tu \in \mathcal{N}_\lambda^+$ (or \mathcal{N}_λ^-) if and only if $m'_u(t) > 0$ (or < 0).

For every $u \in X_0^\alpha(\Omega) \setminus \{0\}$ with $\int_{\Omega} b|u|^{2_\alpha^*} dx > 0$, we let

$$t_{\max}(u) = \left(\frac{(2-q)\|u\|_{X_0^\alpha(\Omega)}^2}{(2_\alpha^* - q) \int_{\Omega} b|u|^{2_\alpha^*} dx} \right)^{\frac{N-2\alpha}{4\alpha}} > 0, \quad (2.14)$$

which leads the following lemma.

Lemma 2.5 *Suppose that $\int_{\Omega} b|u|^{2_\alpha^*} dx > 0$. Then for each $u \in X_0^\alpha(\Omega) \setminus \{0\}$ and $\lambda \in (0, \Lambda_1)$, we have that*

(1) if $\int_{\Omega} a_\lambda |u|^q dx \leq 0$, then there exists a unique $t^- = t^-(u) > t_{\max}(u)$ such that $t^-u \in \mathcal{N}_\lambda^-$ and

$$\Phi_\lambda(t^-u) = \sup_{t \geq 0} \Phi_\lambda(tu). \quad (2.15)$$

(2) if $\int_{\Omega} a_\lambda |u|^q dx > 0$, then there exists a unique $0 < t^+ = t^+(u) < t_{\max}(u) < t^-$ such that $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda^-$ and

$$\Phi_\lambda(t^+u) = \inf_{0 \leq t \leq t_{\max}(u)} \Phi_\lambda(tu), \quad \Phi_\lambda(t^-u) = \sup_{t \geq t^+} \Phi_\lambda(tu). \quad (2.16)$$

Proof. By (2.13), we know t_{\max} is the unique critical point of m_u and m_u is strictly increasing on $(0, t_{\max})$ and strictly decreasing on (t_{\max}, ∞) with $\lim_{t \rightarrow \infty} m_u(t) = -\infty$. Moreover, by the Hölder and Sobolev inequalities, we have that

$$\begin{aligned} m_u(t_{\max}) &\geq \left(\frac{2-q}{2_\alpha^* - q} \right)^{\frac{(N-2\alpha)(2-q)}{4\alpha}} \cdot \left(\frac{2_\alpha^* - 2}{2_\alpha^* - q} \right) S_\alpha^{\frac{N(2-q)}{4\alpha} + \frac{q}{2}} \|u\|_{L^{2_\alpha^*}(\Omega)}^q \\ &\geq \left(\frac{2-q}{2_\alpha^* - q} \right)^{\frac{(N-2\alpha)(2-q)}{4\alpha}} \cdot \left(\frac{2_\alpha^* - 2}{2_\alpha^* - q} \right) S_\alpha^{\frac{N(2-q)}{4\alpha} + \frac{q}{2}} \frac{\int_\Omega a_\lambda |u|^q dx}{\lambda \|a_+\|_{L^{q^*}(\Omega)}} \\ &= \Lambda_1 \lambda^{-1} \int_\Omega a_\lambda |u|^q dx > \int_\Omega a_\lambda |u|^q dx. \end{aligned}$$

Next, we fix $u \in X_0^\alpha(\Omega) \setminus \{0\}$. Suppose that $\int_\Omega a_\lambda |u|^q dx \leq 0$. Then $m_u(t) = \int_\Omega a_\lambda |u|^q$ has unique solution $t^- > t_{\max}$ and $m'_u(t^-) < 0$. Hence h_u has a unique turning point at $t = t^-$ and $h''(t^-) < 0$. Thus $t^-u \in \mathcal{N}_\lambda^-$ and (2.15) holds.

Suppose $\int_\Omega a_\lambda |u|^q dx > 0$. Since $m_u(t_{\max}) > \int_\Omega a_\lambda \|u\|^q dx$, the equation $m_u(t) = \int_\Omega a_\lambda |u|^q$ has exactly two solutions $0 < t^+ < t_{\max}(u) < t^-$ such that $m'_u(t^+) > 0$ and $m'_u(t^-) < 0$. Hence, there are two multiplies of u lying in \mathcal{N}_λ , that is, $t^+u \in \mathcal{N}_\lambda^+$ and $t^-u \in \mathcal{N}_\lambda^-$. Thus h_u has turning points at $t = t^+$ and $t = t^-$ with $h''(t^+) < 0$ and $h''(t^-) < 0$. Thus, h_u is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on (t^-, ∞) . Hence (2.16) holds. \square

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by variational methods. Firstly, by Lemma 2.5, we know that \mathcal{N}_λ^+ and \mathcal{N}_λ^- are non-empty. Moreover, by Lemma 2.4, we can write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ and Lemma 2.1, we can define

$$c_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} \Phi_\lambda(u) \quad \text{and} \quad c_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} \Phi_\lambda(u).$$

Lemma 3.1 (1) For all $\lambda \in (0, \Lambda_1)$, we have $c_\lambda^+ < 0$;

(2) If $\lambda < \Lambda_0 = \frac{1}{2}q\Lambda_1$, then $c_\lambda^- > 0$. In particular, $c_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda} \Phi_\lambda(u)$ for all $\lambda \in (0, \Lambda_0)$.

Proof. (1) Let $u \in \mathcal{N}_\lambda^+$. Then, by (2.8), we have

$$\|u\|_{X_0^\alpha(\Omega)}^2 < \frac{2_\alpha^* - q}{2_\alpha^* - 2} \int_\Omega a_\lambda |u|^q dx.$$

Hence, by (2.5) and Lemma 2.3, we have

$$\begin{aligned}\Phi_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \|u\|_{X_0^\alpha(\Omega)}^2 - \left(\frac{1}{q} - \frac{1}{2_\alpha^*}\right) \int_\Omega a_\lambda |u|^q dx \\ &< -\frac{(2_\alpha^* - q)(2 - q)}{2q \cdot 2_\alpha^*} \int_\Omega a_\lambda |u|^q dx < 0.\end{aligned}$$

Thus, $c_\lambda^+ < 0$.

(2) Let $u \in \mathcal{N}_\lambda^-$. Then, by (2.9), we have

$$\frac{2 - q}{2_\alpha^* - q} \|u\|_{X_0^\alpha(\Omega)}^2 < \int_\Omega b |u|^{2_\alpha^*} dx \leq \int_\Omega |u|^{2_\alpha^*} dx \leq S_\alpha^{-\frac{2_\alpha^*}{2}} \|u\|_{X_0^\alpha(\Omega)}^{2_\alpha^*}$$

and so

$$\|u\|_{X_0^\alpha(\Omega)} > S_\alpha^{\frac{N}{4\alpha}} \left(\frac{2 - q}{2_\alpha^* - q} \right)^{\frac{N - 2\alpha}{4\alpha}}.$$

Therefore, by (2.6), we know

$$\begin{aligned}\Phi_\lambda(u) &\geq \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \|u\|_{X_0^\alpha(\Omega)}^2 - \lambda \left(\frac{1}{q} - \frac{1}{2_\alpha^*}\right) \|a_+\|_{L^{q^*}(\Omega)} S_\alpha^{-q/2} \|u\|_{X_0^\alpha(\Omega)}^q \\ &> \|u\|_{X_0^\alpha(\Omega)}^q \left(\frac{\alpha}{N} S_\alpha^{\frac{N(2-q)}{4\alpha}} \left(\frac{2 - q}{2_\alpha^* - q} \right)^{\frac{(N-2\alpha)(2-q)}{4\alpha}} - \lambda \frac{q - 2_\alpha^*}{q 2_\alpha^*} \|a_+\|_{L^{q^*}(\Omega)} S_\alpha^{-\frac{q}{2}} \right).\end{aligned}$$

Thus, if $\lambda < \Lambda_0 = \frac{q}{2} \Lambda_1$, then $\Phi_\lambda(u) > 0$. This completes the proof. \square

We need the following proposition for the precise description of the Palais-Smale sequence of Φ_λ .

Proposition 3.1 *Each sequence $\{u_n\} \subset \mathcal{N}_\lambda$ that satisfies*

- (1) $\Phi_\lambda(u_n) = c + o(1)$ with $c < c_\lambda^+ + \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}$;
- (2) $\Phi'_\lambda(u_n) = o(1)$ in $(X_0^\alpha(\Omega))^*$

has a convergent subsequence.

The proof of Proposition 3.1 is very similar to Proposition 3.2 in [32], we omit it here.

Next, we establish the existence of a local minimum for Φ_λ on \mathcal{N}_λ^+ .

Theorem 3.1 *For each $0 < \lambda < \Lambda_0$, the functional Φ_λ has a minimizer u_λ^+ in \mathcal{N}_λ^+ satisfying that*

- (1) $\Phi_\lambda(u_\lambda^+) = c_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} \Phi_\lambda(u)$;
- (2) u_λ^+ is a positive solution of (1.1).

Proof. By Lemma 2.1, we know Φ_λ is bounded blow on \mathcal{N}_λ as well as \mathcal{N}_λ^+ . Thus, by Ekeland variational principle [14], there exists $\{u_n\} \subset \mathcal{N}_\lambda^+$ such that it is a $(PS)_{c_\lambda^+}$ -sequence for Φ_λ . Then by Proposition 3.1, there exists a subsequence of $\{u_n\}$ such that $u_n \rightarrow u_\lambda^+$ strongly in $X_0^\alpha(\Omega)$. Moreover, $\Phi_\lambda(|u_\lambda^+|) \leq \Phi_\lambda(u_\lambda^+)$ (see (A.11) in [24]) and $|u_\lambda^+| \in \mathcal{N}_\lambda^+$, by Lemma 2.2, we may assume u_λ^+ is a positive solution of (1.1). \square

Next, we consider a cut-off function $\eta \in C^\infty(\mathbb{R}^N)$ with $0 \leq \eta \leq 1$, $|\nabla \eta| \leq C$, $\eta = 1$ if $|x| \leq r_0/2$ and $\eta = 0$ if $|x| \geq r_0$. For any $z \in M$ (see hypothesis (H_2)), let

$$w_{\varepsilon,z}(x) = \eta(x-z)U_\varepsilon(x-z), \quad (3.1)$$

where U_ε given by (2.4) with $x_0 = 0$. By similar argument as Propositions 21 and 22 in [25], we have that

$$\|w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}^2 = S_\alpha^{\frac{N}{2\alpha}} + O(\varepsilon^{N-2\alpha}) \quad \text{and} \quad \int_\Omega |w_{\varepsilon,z}|^{2^*} dx = S_\alpha^{\frac{N}{2\alpha}} + O(\varepsilon^N) \quad (3.2)$$

hold uniformly for $z \in M$. Hence, by using (3.2) and taking a similar argument as Lemmas 3.1 and 3.2 in [8], we can get the following estimates.

Lemma 3.2 (1) $\int_\Omega b|w_{\varepsilon,z}|^{2^*} dx = S_\alpha^{\frac{N}{2\alpha}} + o(\varepsilon^{\frac{N-2\alpha}{2}})$ uniformly for $z \in M$;
(2) $\int_\Omega |w_{\varepsilon,z}|^q dx = o(\varepsilon^{\frac{N-2\alpha}{2}})$ uniformly for $z \in M$.

Proof. (1) $w_{\varepsilon,z}$ is given by (3.1). We define function $\tilde{b} : \mathbb{R}^N \rightarrow \mathbb{R}$ is an extension of b by $\tilde{b}(x) = b(x)$ if $x \in \bar{\Omega}$ and $\tilde{b}(x) = 0$ if $x \in \mathbb{R}^N \setminus \bar{\Omega}$.

By the definition of U_ε (see (2.4)), we have

$$\begin{aligned} \int_\Omega b|w_{\varepsilon,z}|^{2^*} dx &= \int_{B_{r_0}(z)} b(x)|\eta(x-z)U_\varepsilon(x-z)|^{2^*} dx \\ &= \int_{B_{r_0}(0)} b(x+z)|\eta(x)U_\varepsilon(x)|^{2^*} dx \\ &= \int_{B_{r_0}(0)} \frac{\varepsilon^N S_\alpha^{\frac{N}{2\alpha}} \kappa^{2^*}}{\left(|x|^2 + \varepsilon^2 S_\alpha^{\frac{1}{\alpha}} \mu^2\right)^N} \tilde{b}(x+z) \eta^{2^*}(x) dx \\ &:= \int_{B_{r_0}(0)} \frac{C_0 \varepsilon^N}{(|x|^2 + C_1 \varepsilon^2)^N} \tilde{b}(x+z) \eta^{2^*}(x) dx, \end{aligned}$$

where $C_0 = S_\alpha^{\frac{N}{2\alpha}} \kappa^{2^*}$ and $C_1 = S_\alpha^{\frac{1}{\alpha}} \mu^2$.

Next, by assumption (H_2) and $b(z) = 1$ since $z \in M$, we can see that

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*_\alpha} dx - \int_{\Omega} b|w_{\varepsilon,z}|^{2^*_\alpha} dx \\
&= \int_{\mathbb{R}^N} \frac{C_0 \varepsilon^N}{(|x|^2 + C_1 \varepsilon^2)^N} \left(1 - \tilde{b}(x+z) \eta^{2^*_\alpha}(x)\right) dx \\
&= \int_{\mathbb{R}^N \setminus B_{\frac{r_0}{2}}(0)} \frac{C_0 \varepsilon^N}{(|x|^2 + C_1 \varepsilon^2)^N} \left(1 - \tilde{b}(x+z) \eta^{2^*_\alpha}(x)\right) dx \\
&\quad + \int_{B_{\frac{r_0}{2}}(0)} \frac{C_0 \varepsilon^N}{(|x|^2 + C_1 \varepsilon^2)^N} \left(1 - \tilde{b}(x+z) \eta^{2^*_\alpha}(x)\right) dx \\
&\leq C_0 \varepsilon^N \int_{\mathbb{R}^N \setminus B_{\frac{r_0}{2}}(0)} \frac{1}{|x|^{2N}} dx + D_0 C_0 \varepsilon^N \int_{B_{\frac{r_0}{2}}(0)} \frac{|x|^\rho}{(|x|^2 + C_1 \varepsilon^2)^N} dx \\
&\leq C_0 \varepsilon^N \int_{\frac{r_0}{2}}^\infty r^{-N-1} dr + D_0 C_0 \varepsilon^N \int_0^{\frac{r_0}{2}} \frac{r^{\rho+N-1}}{(r^2 + C_1 \varepsilon^2)^N} dr \\
&\leq O(\varepsilon^N) + D_0 C_0 \varepsilon^N \int_0^\varepsilon \frac{r^{\rho+N-1}}{(r^2 + C_1 \varepsilon^2)^N} dr + D_0 C_0 \varepsilon^N \int_\varepsilon^{\frac{r_0}{2}} \frac{r^{\rho+N-1}}{(r^2 + C_1 \varepsilon^2)^N} dr \\
&\leq O(\varepsilon^N) + D_1 C_0 \varepsilon^{-N} \int_0^\varepsilon r^{\rho+N-1} dr + D_2 C_0 \varepsilon^N \int_\varepsilon^{\frac{r_0}{2}} r^{\rho-N-1} dr \\
&= \begin{cases} O(\varepsilon^N) + O(\varepsilon^\rho) & \text{if } \rho \neq N, \\ O(\varepsilon^N) + C_2 \varepsilon^N |\ln \varepsilon| & \text{if } \rho = N. \end{cases}
\end{aligned}$$

This implies that

$$\int_{\Omega} b|w_{\varepsilon,z}|^{2^*_\alpha} dx = S_\alpha^{\frac{N}{2\alpha}} + o(\varepsilon^{\frac{N-2\alpha}{2}})$$

uniformly for $z \in M$ since $\rho > (N - 2\alpha)/2$ and $\int_{\mathbb{R}^N} |U_\varepsilon|^{2^*_\alpha} dx = S_\alpha^{\frac{N}{2\alpha}}$.

(2) Since

$$\begin{aligned}
\int_{\Omega} |w_{\varepsilon,z}|^q dx &= \int_{B_{r_0}(0)} \eta^q(x) U_\varepsilon^q(x) dx \\
&= \int_{B_{r_0}(0)} \eta^q(x) \frac{C_0 \varepsilon^{\frac{q(N-2\alpha)}{2}}}{(|x|^2 + C_1 \varepsilon^2)^{\frac{q(N-2\alpha)}{2}}} dx \\
&= \int_{B_\varepsilon(0)} \eta^q(x) \frac{C_0 \varepsilon^{\frac{q(N-2\alpha)}{2}}}{(|x|^2 + C_1 \varepsilon^2)^{\frac{q(N-2\alpha)}{2}}} dx \\
&\quad + \int_{B_{r_0}(0) \setminus B_\varepsilon(0)} \eta^q(x) \frac{C_0 \varepsilon^{\frac{q(N-2\alpha)}{2}}}{(|x|^2 + C_1 \varepsilon^2)^{\frac{q(N-2\alpha)}{2}}} dx
\end{aligned}$$

$$\begin{aligned}
&\leq C_3 \int_{B_\varepsilon(0)} \frac{\varepsilon^{\frac{q(N-2\alpha)}{2}}}{\varepsilon^{q(N-2\alpha)}} dx + \int_{B_{r_0}(0) \setminus B_\varepsilon(0)} \frac{C_0 \varepsilon^{\frac{q(N-2\alpha)}{2}}}{|x|^{q(N-2\alpha)}} dx \\
&= \begin{cases} C_4 \varepsilon^{\frac{(N-2\alpha)(2-q)+4\alpha}{2}} + C_5 \varepsilon^{\frac{q(N-2\alpha)}{2}} & \text{if } q \neq \frac{N}{N-2\alpha}, \\ C_3 \varepsilon^{\frac{(N-2\alpha)(2-q)+4\alpha}{2}} + C_6 \varepsilon^{\frac{q(N-2\alpha)}{2}} + C_0 \varepsilon^{\frac{q(N-2\alpha)}{2}} |\ln \varepsilon| & \text{if } q = \frac{N}{N-2\alpha} \end{cases}
\end{aligned}$$

for all $z \in M$. Hence, we have that

$$\int_{\Omega} |w_{\varepsilon,z}|^q dx = o(\varepsilon^{\frac{N-2\alpha}{2}})$$

uniformly for $z \in M$, where we have used the fact $1 < q < (N+2\alpha)(N-2\alpha)$.

□

Next, we have the following result.

Lemma 3.3 *Let Λ_0 as defined in Lemma 3.1, then, for $\lambda < \Lambda_0$,*

$$\sup_{t \geq 0} \Phi_\lambda(u_\lambda^+ + tw_{\varepsilon,z}) < d_\lambda := c_\lambda^+ + \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}$$

uniformly for $z \in M$.

Proof. The proof this lemma is very similar to Lemma 3.2 in [8]. As Lemma 3.2 in [8], we first can get the following inequality

$$\Phi_\lambda(u_\lambda^+ + tw_{\varepsilon,z}) \leq \Phi_\lambda(u_\lambda^+) + J_\lambda(tw_{\varepsilon,z}),$$

where

$$J_\lambda(v) = \frac{1}{2} \|v\|_{X_0^\alpha(\Omega)}^2 + C \int_{\Omega} v^q dx - \frac{1}{2_\alpha^*} \int_{\Omega} b[(u_\lambda^+ + v)^{2_\alpha^*} - (u_\lambda^+)^{2_\alpha^*} - 2_\alpha^+(u_\lambda^+)^{2_\alpha^* - 1}] dx.$$

Then, by Theorem 3.1 (i), we just need to prove that

$$\sup_{t \geq 0} J_\lambda(tw_{\varepsilon,z}) < \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}$$

uniformly for $z \in M$. Applying Lemma 3.2 and following a similar argument as Lemma 3.2 in [8], we can obtain that there exists $t_0 > 0$ and a sufficiently small ε_0 such that

$$J_\lambda(tw_{\varepsilon,z}) \leq 0 < \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}} \quad \text{for all } t \in [t_0, \infty), \quad z \in M \text{ and } 0 < \varepsilon < \varepsilon_0,$$

and

$$\max_{t \in [0, t_0]} J_\lambda(tw_{\varepsilon,z}) < \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}.$$

This completes the proof. □

Next, by using Lemma 3.3, we can find a positive solution in \mathcal{N}_λ^- if $\lambda < \Lambda_0$.

Theorem 3.2 *Let $\Lambda_0 > 0$ as defined in Lemma 3.1, Then, for each $\lambda < \Lambda_0$, equation (1.1) has a positive solution $u_\lambda^- \in \mathcal{N}_\lambda^-$.*

Proof. We first show that $c_\lambda^- \leq c_\lambda^+ + \frac{\alpha}{N} S_\alpha^{N/2\alpha}$ and thus we can apply Proposition 3.1 to obtain a solution. Here we adopt the method of Tarantello [26] and Wu [32]. By Lemma 2.5, we know, for very $u \in X_0^\alpha(\Omega) \setminus \{0\}$, that there exists a unique $t^- = t^-(u) > 0$ such that $t^-(u)u \in \mathcal{N}_\lambda^-$. So we claim that

$$\mathcal{N}_\lambda^- = \left\{ u \in X_0^\alpha(\Omega) \setminus \{0\} \mid \frac{1}{\|u\|_{X_0^\alpha(\Omega)}} t^- \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) = 1 \right\}.$$

In fact, for $u \in \mathcal{N}_\lambda^-$, let $w = \frac{u}{\|u\|_{X_0^\alpha(\Omega)}}$. Then there exists a unique $t^-(w) > 0$ such that $t^-(w)w \in \mathcal{N}_\lambda^-$ or $\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} t^- \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) \in \mathcal{N}_\lambda^-$. Since $u \in \mathcal{N}_\lambda^-$, we have $\frac{1}{\|u\|_{X_0^\alpha(\Omega)}} t^- \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) = 1$. This implies

$$\mathcal{N}_\lambda^- \subset \left\{ u \in X_0^\alpha(\Omega) \setminus \{0\} \mid \frac{1}{\|u\|_{X_0^\alpha(\Omega)}} t^- \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) = 1 \right\}.$$

Conversely, let $u \in X_0^\alpha(\Omega) \setminus \{0\}$ such that $\frac{1}{\|u\|_{X_0^\alpha(\Omega)}} t^- \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) = 1$. Then

$$u = t^- \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) \frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \in \mathcal{N}_\lambda^-.$$

Next, we let

$$A_1 = \left\{ u \in X_0^\alpha(\Omega) \setminus \{0\} \mid \frac{1}{\|u\|_{X_0^\alpha(\Omega)}} t^- \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) > 1 \right\} \cup \{0\};$$

$$A_2 = \left\{ u \in X_0^\alpha(\Omega) \setminus \{0\} \mid \frac{1}{\|u\|_{X_0^\alpha(\Omega)}} t^- \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) < 1 \right\}.$$

Then \mathcal{N}_λ^- disconnects $X_0^\alpha(\Omega)$ in two connected components A_1 and A_2 . Clearly, $X_0^\alpha(\Omega) \setminus \mathcal{N}_\lambda^- = A_1 \cup A_2$ and $\mathcal{N}_\lambda^+ \subset A_1$. Indeed, for $u \in \mathcal{N}_\lambda^+$, there exist unique $t^- \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) > 0$ and $t^+ \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) > 0$ such that

$$t^+ \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) < t_{\max} < t^- \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right)$$

and $t^+ \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) \frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \in \mathcal{N}_\lambda^+$. Since $u \in \mathcal{N}_\lambda^+$, we have that

$$\frac{1}{\|u\|_{X_0^\alpha(\Omega)}} t^+ \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) = 1.$$

Therefore,

$$t^- \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) > t^+ \left(\frac{u}{\|u\|_{X_0^\alpha(\Omega)}} \right) = \|u\|_{X_0^\alpha(\Omega)}.$$

This implies $\mathcal{N}_\lambda^+ \subset A_1$.

Next, we claim that there exists a $l_0 > 0$ such that $u_\lambda^+ + l_0 w_{\varepsilon,z} \in A_2$. Firstly, we find a constant $C_{19} > 0$ such that $0 < t^- \left(\frac{u_\lambda^+ + l w_{\varepsilon,z}}{\|u_\lambda^+ + l w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}} \right) < C_{19}$ for each $l > 0$. Otherwise, there exists a sequence $\{l_n\}$ such that $l_n \rightarrow \infty$ and $t^- \left(\frac{u_\lambda^+ + l_n w_{\varepsilon,z}}{\|u_\lambda^+ + l_n w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}} \right) \rightarrow \infty$. Let $v_n = \frac{u_\lambda^+ + l_n w_{\varepsilon,z}}{\|u_\lambda^+ + l_n w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}}$. Since $t^-(v_n)v_n \in \mathcal{N}_\lambda^-$, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \int_\Omega b|v_n|^{2_\alpha^*} dx &= \frac{1}{\|u_\lambda^+ + l_n w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}^{2_\alpha^*}} \int_\Omega b(x) |u_\lambda^+ + l_n w_{\varepsilon,z}|^{2_\alpha^*} dx \\ &= \frac{1}{\left\| \frac{u_\lambda^+}{l_n} + w_{\varepsilon,z} \right\|_{X_0^\alpha(\Omega)}^{2_\alpha^*}} \int_\Omega b(x) \left| \frac{u_\lambda^+}{l_n} + w_{\varepsilon,z} \right|^{2_\alpha^*} dx \\ &\rightarrow \frac{1}{\|w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}^{2_\alpha^*}} \int_\Omega |w_{\varepsilon,z}|^{2_\alpha^*} dx > 0 \end{aligned}$$

as $n \rightarrow \infty$ and

$$\begin{aligned} \Phi_\lambda(t^-(v_n)v_n) &= \frac{(t^-(v_n))^2}{2} \|v_n\|_{X_0^\alpha(\Omega)}^2 - \frac{t^-(v_n)^q}{q} \int_\Omega a_\lambda |v_n|^q dx - \frac{t^-(v_n)^{2_\alpha^*}}{2_\alpha^*} \int_\Omega b|v_n|^{2_\alpha^*} dx \\ &\rightarrow -\infty, \end{aligned}$$

as $n \rightarrow \infty$, which contradicts the fact that Φ_λ is bounded below on N_λ^- .

Now, we let

$$l_0 = \frac{\left| C_{19}^2 - \|u_\lambda^+\|_{X_0^\alpha(\Omega)}^2 \right|^{\frac{1}{2}}}{\|w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}} + 1.$$

Then,

$$\begin{aligned} \|u_\lambda^+ + l_0 w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}^2 &= \|u_\lambda^+\|_{X_0^\alpha(\Omega)}^2 + l_0^2 \|w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}^2 + 2l_0 \langle u_\lambda^+, w_{\varepsilon,z} \rangle \\ &> \|u_\lambda^+\|_{X_0^\alpha(\Omega)}^2 + \left| C_{19}^2 - \|u_\lambda^+\|_{X_0^\alpha(\Omega)}^2 \right| \\ &> C_{19}^2 > \left[t^- \left(\frac{u_\lambda^+ + l w_{\varepsilon,z}}{\|u_\lambda^+ + l w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}} \right) \right]^2 \end{aligned}$$

and this implies $u_\lambda^+ + l_0 w_{\varepsilon,z} \in A_2$.

Next, we define a path

$$\gamma(s) = u_\lambda^+ + s l_0 w_{\varepsilon,z}$$

for $s \in [0, 1]$. Then $\gamma(0) = u_\lambda^+ \in \mathcal{N}_\lambda^+ \subset A_1$ and $\gamma(1) = u_\lambda^+ + l_0 w_{\varepsilon,z} \in A_2$. Then there exists a $s_0 \in (0, 1)$ such that

$$\gamma(s_0) = u_\lambda^+ + s_0 l_0 w_{\varepsilon,z} \in \mathcal{N}_\lambda^-.$$

Therefore, by Lemma 3.2, we know that

$$c_\lambda^- \leq \Phi_\lambda(u_\lambda^+ + s_0 l_0 w_{\varepsilon,z}) < c_\lambda^+ + \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}.$$

Similarly, by the Ekeland variation principle (see [14]) since Φ_λ is bounded blow on \mathcal{N}_λ as well as on \mathcal{N}_λ^- , such a minimizing sequence $\{u_n\} \in \mathcal{N}_\lambda^-$ for Φ_λ can be established such that

$$\Phi_\lambda(u_n) = c_\lambda^- + o(1) \quad \text{and} \quad \Phi'_\lambda(u_n) = o(1) \quad \text{in } (X_0^\alpha(\Omega))^*.$$

By Proposition 3.1, there exists a subsequence $\{u_n\}$ and $u_\lambda^- \in \mathcal{N}_\lambda^-$ such that $u_n \rightarrow u_\lambda^-$ strongly in $X_0^\alpha(\Omega)$, $\Phi_\lambda(u_\lambda^-) = c_\lambda^-$ and u_λ^- is a positive solution of equation (1.1) by a similar argument as in Theorem 3.1. \square

Proof of Theorem 1.1. Together with Theorems 3.1 and 3.2, we obtain Theorem 1.1. \square

4 Proof of Theorem 1.2

We first consider the following critical problem

$$\begin{cases} (-\Delta)^\alpha u = |u|^{2_\alpha^*-1} u & \text{in } \Omega, \\ u \in X_0^\alpha(\Omega), \end{cases}$$

and, accordingly, the energy functional Φ^∞ in $X_0^\alpha(\Omega)$ is

$$\Phi^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy - \frac{1}{2_\alpha^*} \int_\Omega |u|^{2_\alpha^*} dx.$$

It is easy to check (using the definition of S_α) that

$$\inf_{u \in \mathcal{N}^\infty(\Omega)} \Phi^\infty(u) = \inf_{u \in \mathcal{N}^\infty(\mathbb{R}^N)} \Phi^\infty(u) = \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}},$$

where

$$\mathcal{N}^\infty(\mathbb{R}^N) = \{u \in \dot{H}^\alpha(\mathbb{R}^N) \setminus \{0\} \mid \langle (\Phi^\infty)'(u), u \rangle = 0\}$$

and

$$\mathcal{N}^\infty(\Omega) = \{u \in X_0^\alpha(\Omega) \setminus \{0\} \mid \langle (\Phi^\infty)'(u), u \rangle = 0\}$$

is the Nehari manifold. When $\lambda = 0$, we write Φ_λ (resp. \mathcal{N}_λ) as Φ_0 (resp. \mathcal{N}_0). Then, we have the following results.

Lemma 4.1 *We have that*

$$\inf_{u \in \mathcal{N}_0} \Phi_0(u) = \inf_{u \in \mathcal{N}^\infty} \Phi^\infty(u) = \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}.$$

Moreover, equation (1.1) with $\lambda = 0$ does not admits any positive solution u_0 , for which $\Phi_0(u_0) = \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}$.

Proof. By Lemma 2.5, there exists a unique $t_0(w_{\varepsilon,z})$ such that $t_0(w_{\varepsilon,z})w_{\varepsilon,z} \in \mathcal{N}_0$ for all $\varepsilon > 0$, that is,

$$t_0(w_{\varepsilon,z}) > t_{\max}(w_{\varepsilon,z}) = \left(\frac{(2-q)\|w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}^2}{(2_\alpha^* - q) \int_\Omega b|w_{\varepsilon,z}|^{2_\alpha^*} dx} \right)^{\frac{N-2\alpha}{4\alpha}} \quad (4.1)$$

and

$$\|t_0(w_{\varepsilon,z})w_{\varepsilon,z}\|_{X_0^\alpha(\Omega)}^2 = \int_\Omega a_- |t_0(w_{\varepsilon,z})w_{\varepsilon,z}|^q dx + \int_\Omega b |t_0(w_{\varepsilon,z})w_{\varepsilon,z}|^{2_\alpha^*} dx. \quad (4.2)$$

Moreover, by Lemma 3.2 and the boundedness of a_- , we have

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega b |w_{\varepsilon,z}|^{2_\alpha^*} dx = S_\alpha^{\frac{N}{2\alpha}} \quad (4.3)$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega a_- |w_{\varepsilon,z}|^q dx = 0 \quad (4.4)$$

uniformly for $z \in M$.

Hence, by (3.2) and (4.1)-(4.4), we have

$$\lim_{\varepsilon \rightarrow 0} t_0(w_{\varepsilon,z}) = 1$$

uniformly for $z \in M$. Therefore

$$\inf_{u \in \mathcal{N}_0} \Phi_0(u) \leq \Phi_0(t_0(w_{\varepsilon,z})w_{\varepsilon,z}) \rightarrow \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}$$

as $\varepsilon \rightarrow 0$. So

$$\inf_{u \in \mathcal{N}_0} \Phi_0(u) \leq \inf_{u \in \mathcal{N}^\infty} \Phi^\infty(u) = \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}.$$

Conversely, let $u \in \mathcal{N}_0$. By Lemma 2.5 and the uniqueness, we have $\Phi_0(u) = \sup_{t \geq 0} \Phi_0(tu)$. Moreover, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}^\infty$. Therefore,

$$\Phi_0(u) \geq \Phi_0(t_u u) \geq \Phi^\infty(t_u u) \geq \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}.$$

This implies that $\inf_{u \in \mathcal{N}_0} \Phi_0(u) \geq \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}$. Consequently,

$$\inf_{u \in \mathcal{N}_0} \Phi_0(u) = \inf_{u \in \mathcal{N}^\infty} \Phi^\infty(u) = \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}.$$

Next, we prove that problem (1.1) does not admit any solution u_0 satisfying $\Phi_0(u_0) = \inf_{u \in \mathcal{N}_0} \Phi_0(u)$. We prove it by contradiction. Assume that there exists $u_0 \in \mathcal{N}_0$ and satisfying $\Phi_0(u_0) = \inf_{u \in \mathcal{N}_0} \Phi_0(u)$. As in the proof of Theorem 3.1, we may assume u_0 is a positive solution. By Lemma 2.5 gives that $\Phi_0(u_0) = \sup_{t \geq 0} \Phi_0(tu_0)$, leading to the conclusion that there must exist a unique $t_{u_0} > 0$ such that $t_{u_0} u_0 \in \mathcal{N}^\infty$ and

$$\begin{aligned} \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}} = \inf_{u \in \mathcal{N}_0} \Phi_0(u) &= \Phi_0(u_0) \\ &\geq \Phi_0(t_{u_0} u_0) \\ &\geq \Phi^\infty(t_{u_0} u_0) + \frac{1}{2_\alpha^*} \int_\Omega (1 - b(x)) |t_{u_0} u_0|^{2_\alpha^*} dx \\ &> \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}} + \frac{1}{2_\alpha^*} \int_\Omega (1 - b(x)) |t_{u_0} u_0|^{2_\alpha^*} dx. \end{aligned}$$

This implies that

$$\int_\Omega (1 - b(x)) |u_0|^{2_\alpha^*} dx < 0,$$

which contradicts the fact that $b \leq 1$ in Ω . This completes the proof. \square

By using Lemma 4.1, we have the following result.

Lemma 4.2 *Assume that $\{u_n\}$ is a minimizing sequence for Φ_0 in \mathcal{N}_0 . Then,*

- (1) $\int_\Omega a_- |u_n|^q dx = o(1);$
- (2) $\int_\Omega (1 - b) |u_n|^{2_\alpha^*} dx = o(1).$

Moreover, $\{u_n\}$ is a $(PS)_{\frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}}$ -sequence for Φ^∞ in $X_0^\alpha(\Omega)$.

Proof. For each n , there is a unique $t_n > 0$ such that $t_n u_n \in \mathcal{N}^\infty$, that is,

$$t_n^2 \|u_n\|_{X_0^\alpha(\Omega)}^2 = t_n^{2^*} \int_{\Omega} |u_n|^{2^*} dx.$$

By Lemma 2.5,

$$\begin{aligned} \Phi_0(u_n) \geq \Phi_0(t_n u_n) &= \Phi^\infty(t_n u_n) - \frac{t_n^q}{q} \int_{\Omega} a_- |u_n|^q dx + \frac{t_n^{2^*}}{2_\alpha^*} \int_{\Omega} (1-b) |u_n|^{2^*} dx \\ &\geq \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}} - \frac{t_n^q}{q} \int_{\Omega} a_- |u_n|^q dx + \frac{t_n^{2^*}}{2_\alpha^*} \int_{\Omega} (1-b) |u_n|^{2^*} dx. \end{aligned}$$

Since $\Phi_0(u_n) = \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}} + o(1)$ by Lemma 3.2, we have

$$\frac{t_n^q}{q} \int_{\Omega} a_- |u_n|^q dx = o(1)$$

and

$$\frac{t_n^{2^*}}{2_\alpha^*} \int_{\Omega} (1-b) |u_n|^{2^*} dx = o(1).$$

Next, we prove that there exists $c_0 > 0$ such that $t_n > c_0$ for all n . Suppose the contrary. Then we may assume $t_n \rightarrow 0$ as $n \rightarrow \infty$. Since $\Phi_0(u_n) = \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}} + o(1)$ (see Lemma 4.1), we know that $\|u_n\|_{X_0^\alpha(\Omega)}$ is uniformly bounded by Lemma 2.1. Hence, $\|t_n u_n\|_{X_0^\alpha(\Omega)} \rightarrow 0$ and

$$\begin{aligned} \Phi^\infty(t_n u_n) &= \frac{t_n^2}{2} \|u_n\|_{X_0^\alpha(\Omega)}^2 - \frac{t_n^{2^*}}{2_\alpha^*} \int_{\Omega} |u_n|^{2^*} dx \\ &= \frac{\alpha}{N} \|t_n u_n\|_{X_0^\alpha(\Omega)}^2 \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. This contradict $\Phi^\infty(t_n u_n) \geq \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}} > 0$.

Therefore,

$$\int_{\Omega} a_- |u_n|^q dx = o(1)$$

and

$$\int_{\Omega} (1-b) |u_n|^{2^*} dx = o(1).$$

This implies

$$\|u_n\|_{X_0^\alpha(\Omega)}^2 = \int_{\Omega} |u_n|^{2^*} dx + o(1)$$

and

$$\Phi^\infty(u_n) = \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}} + o(1).$$

Then, by a similar argument as Lemma 7 in [29], we have $\{u_n\}$ is a $(PS)_{\frac{\alpha}{N}S_\alpha^{\frac{N}{2\alpha}}}$ -sequence for Φ^∞ in $X_0^\alpha(\Omega)$. \square

Next, for a positive d , we consider the filtration of the Nehari manifold \mathcal{N}_0 with

$$\mathcal{N}_0(d) = \left\{ u \in \mathcal{N}_0 \mid \Phi_0(u) \leq \frac{\alpha}{N}S_\alpha^{\frac{N}{2\alpha}} + d \right\}$$

and the function

$$F(u) = \frac{\int_\Omega x|u|^{2_\alpha^*} dx}{\int_\Omega |u|^{2_\alpha^*} dx}.$$

With these notations, we have the following result.

Lemma 4.3 *For each $0 < \delta < r_0$, there exists $d_\delta > 0$ such that*

$$F(u) \in M_\delta \quad \text{for all } u \in \mathcal{N}_0(d_\delta).$$

Proof. We prove it by contradiction. Assume that there exists a sequence $\{u_n\} \subset \mathcal{N}_0$ and $\delta_0 < r_0$ such that $\Phi_0(u_n) \leq \frac{\alpha}{N}S_\alpha^{\frac{N}{2\alpha}} + o(1)$ and $F(u_n) \notin M_{\delta_0}$ for all n .

By Lemma 4.2, we know that $\{u_n\}$ is also a $(PS)_{\frac{\alpha}{N}S_\alpha^{\frac{N}{2\alpha}}}$ -sequence for Φ^∞ in $X_0^\alpha(\Omega)$. Clearly, $\|u_n\|_{X_0^\alpha(\Omega)}$ is bounded and thus there exists a subsequence $\{u_n\}$ and $u_0 \in X_0^\alpha(\Omega)$ such that $u_n \rightharpoonup u_0$ in $X_0^\alpha(\Omega)$. Since Ω is bounded, we have $u_0 \equiv 0$. Therefore, by the concentration-compactness principle (see Theorem 6 in [20]), there exists two sequences $\{x_n\} \subset \Omega$ and $\{R_n\} \subset \mathbb{R}^+$ with $x_0 \in \bar{\Omega}$ such that $x_n \rightarrow x_0$ and $R_n \rightarrow \infty$ and

$$\|u_n(x) - R_n^{\frac{N-2\alpha}{2}} u_0(R_n(x - x_n))\|_{L^{2_\alpha^*}(\mathbb{R}^N)} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where u_0 is defined as (2.2).

Therefore,

$$\begin{aligned} F(u_n) &= \frac{\int_\Omega x|u_n|^{2_\alpha^*} dx}{\int_\Omega |u_n|^{2_\alpha^*} dx} \\ &= \frac{\int_\Omega x \left| R_n^{\frac{N-2\alpha}{2}} u_0(R_n(x - x_n)) \right|^{2_\alpha^*} dx}{\int_\Omega \left| R_n^{\frac{N-2\alpha}{2}} u_0(R_n(x - x_n)) \right|^{2_\alpha^*} dx} + o(1) \\ &= \frac{\int_\Omega \left(\frac{x}{R_n} + x_n \right) |u_0(x)|^{2_\alpha^*} dx}{\int_\Omega |u_0(x)|^{2_\alpha^*} dx} + o(1) \\ &= x_0 + o(1). \end{aligned}$$

Next, we show that $x_0 \in M_{\delta_0}$. Since $\{u_n\}$ is a minimizing sequence for Φ_0 in \mathcal{N}_0 , by Lemma 4.2,

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_{\Omega} (1-b)|u_n|^{2^*_{\alpha}} dx = \lim_{n \rightarrow \infty} \int_{\Omega} (1-b)|R_n^{\frac{N-2\alpha}{2}} U_0(R_n(x-x_n))|^{2^*_{\alpha}} dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} \left(1-b\left(\frac{x}{R_n} + x_n\right)\right) |U_0(x)|^{2^*_{\alpha}} dx \\ &= (1-b(x_0)) S_{\alpha}^{\frac{N}{2\alpha}}. \end{aligned}$$

This implies that $b(x_0) = \max_{x \in \bar{\Omega}} b(x) \equiv 1$ and thus $x_0 \in M$, which contradicts our assumption. This completes the proof. \square

We now proceed to consider the filtration of the manifold $\mathcal{N}_{\lambda}^{-}$ with

$$\mathcal{N}_{\lambda}(c) = \{u \in \mathcal{N}_{\lambda}^{-} \mid \Phi_{\lambda}(u) \leq c\}.$$

We can prove that

Lemma 4.4 *For each $0 < \delta < r_0$, there exists $0 < \Lambda_{\delta} \leq \Lambda_0$ such that, for $\lambda < \Lambda_{\delta}$, we have*

$$F(u) \in M_{\delta} \quad \text{for all } u \in \mathcal{N}_{\lambda}(d_{\lambda}),$$

where d_{λ} is defined as in Lemma 3.3.

Proof. For $u \in \mathcal{N}_{\lambda}(d_{\lambda})$ and thus $u \in \mathcal{N}_{\lambda}^{-}$, by (2.9) and Lemma 2.5, there exists a unique $t_u > t_{\max}(u)$ such that $t_u u \in \mathcal{N}_0$. Therefore, by the Hölder and Sobolev inequalities,

$$\begin{aligned} \Phi_{\lambda}(u) &= \sup_{t \geq t_{\max}(u)} \Phi_{\lambda}(tu) \\ &\geq \Phi_{\lambda}(t_u u) \\ &= \Phi_0(t_u u) - \frac{\lambda t_u^q}{q} \int_{\Omega} a_+ |u|^q dx \\ &\geq \Phi_0(t_u u) - \frac{\lambda t_u^q}{q} \|a_+\|_{L^{q^*}(\Omega)} S_{\alpha}^{-\frac{q}{2}} \|u\|_{X_0^{\alpha}(\Omega)}^q. \end{aligned} \tag{4.5}$$

Next, we prove that there exists a positive constant κ_0 independent of u such that $t_u \leq \kappa_0$. In fact, by (2.9) and the Sobolev inequality,

$$\|u\|_{X_0^{\alpha}(\Omega)}^2 < \frac{2^*_{\alpha} - q}{2 - q} \int_{\Omega} b|u|^{2^*_{\alpha}} dx \leq \frac{2^*_{\alpha} - q}{2 - q} S_{\alpha}^{-\frac{2^*_{\alpha}}{2}} \|u\|_{X_0^{\alpha}(\Omega)}^{2^*_{\alpha}}, \tag{4.6}$$

and then,

$$\|u\|_{X_0^{\alpha}(\Omega)} \geq \left(\frac{(2-q) S_{\alpha}^{\frac{2^*_{\alpha}}{2}}}{2^*_{\alpha} - q} \right)^{\frac{N-2\alpha}{4\alpha}}. \tag{4.7}$$

Without loss of generality, we may assume that $t_u \geq 1$. Since

$$t_u^{2^*} \int_{\Omega} b|u|^{2^*} dx = t_u^2 \|u\|_{X_0^\alpha(\Omega)}^2 - t_u^q \int_{\Omega} a_- |u|^q dx \leq t_u^2 \left(\|u\|_{X_0^\alpha(\Omega)}^2 + \int_{\Omega} |a_-| |u|^q \right),$$

we have

$$t_u \leq \left(\frac{\|u\|_{X_0^\alpha(\Omega)}^2 + \int_{\Omega} |a_-| |u|^q}{\int_{\Omega} b|u|^{2^*} dx} \right)^{\frac{N-2\alpha}{4\alpha}}. \quad (4.8)$$

Hence, by (4.6)-(4.8) and the Hölder and Sobolev inequalities,

$$\begin{aligned} t_u &\leq \left[\frac{2_\alpha^* - q}{2 - q} \left(1 + \frac{\int_{\Omega} |a_-| |u|^q}{\|u\|_{X_0^\alpha(\Omega)}^2} \right) \right]^{\frac{N-2\alpha}{4\alpha}} \\ &\leq \left[\frac{2_\alpha^* - q}{2 - q} \left(1 + \frac{\|a_-\|_{L^{q^*}(\Omega)}}{S_\alpha^{\frac{q}{2}} \|u\|_{X_0^\alpha(\Omega)}^{2-q}} \right) \right]^{\frac{N-2\alpha}{4\alpha}} \\ &\leq \left[\frac{2_\alpha^* - q}{2 - q} \left(1 + \|a_-\|_{L^{q^*}(\Omega)} \left(\frac{2_\alpha^* - q}{(2-q) S_\alpha^{\frac{2_\alpha^*-q}{2-q}}} \right)^{\frac{2-q}{2_\alpha^*-q}} \right) \right]^{\frac{N-2\alpha}{4\alpha}} \\ &= \kappa_0. \end{aligned} \quad (4.9)$$

Substituting (4.9) into (4.5), we have that

$$\Phi_0(t_u u) \leq \Phi_\lambda(u) + \frac{\lambda t_u^q}{q} \|a_+\|_{L^{q^*}(\Omega)} S_\alpha^{-\frac{q}{2}} \|u\|_{X_0^\alpha(\Omega)}^q \leq d\lambda + \frac{\lambda \kappa_0^q}{q} \|a_+\|_{L^{q^*}(\Omega)} S_\alpha^{-\frac{q}{2}} \|u\|_{X_0^\alpha(\Omega)}^q.$$

Since $\Phi_\lambda(u) \leq d_\lambda < \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}$, by the proof of Lemma 2.1, for each $0 < \lambda < \Lambda_0$, there exists a positive constant c_0 independent of λ such that $\|u\|_{X_0^\alpha(\Omega)} \leq c_0$ for all $u \in \mathcal{N}_\lambda(d_\lambda)$. Hence,

$$\Phi_0(t_u u) \leq d_\lambda + \frac{\lambda \kappa_0^q}{q} \|a_+\|_{L^{q^*}(\Omega)} S_\alpha^{-\frac{q}{2}} c_0^q.$$

Let $d_\delta > 0$ be as in Lemma 4.3. Then there exists $0 < \Lambda_\delta \leq \Lambda_0$ such that, for $\lambda < \Lambda_\delta$,

$$\Phi_0(t_u u) \leq \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}} + d_\delta.$$

for all $u \in \mathcal{N}_\lambda(d_\lambda)$. By Lemma 4.3, we have $t_u u \in \mathcal{N}_0(d_\delta)$ and

$$F(u) = \frac{\int_{\Omega} x |t_u u|^{2^*} dx}{\int_{\Omega} |t_u u|^{2^*} dx} = F(t_u u) \in M_\delta$$

for all $u \in \mathcal{N}_\lambda^-(d_\lambda)$. This completes the proof. \square

We recall a multiplicity result for critical points involving Ljusternik-Schnirelman category, which shall apply in proving Theorem 1.2 (for the proof e.g., see [16]).

Theorem 4.1 *Let \mathcal{M} be a $C^{1,1}$ complete Riemannian manifold (modelled on a Hilbert space) and assume $\Psi \in C^1(\mathcal{M}, \mathbb{R})$ bounded from below. Let $-\infty < \inf_{\mathcal{M}} \Psi < \sigma < \tau < \infty$. Suppose that Ψ satisfies (PS)-condition on the sublevel $\{u \in \mathcal{M} \mid \Psi(u) \leq \tau\}$ and σ is not a critical level for Ψ . Then there exists at least $\text{cat}_{\Psi^\sigma}(\Psi^\sigma)$ critical points of Ψ in Ψ^σ , where $\Psi^\sigma = \{u \in \mathcal{M} \mid \Psi(u) \leq \sigma\}$.*

Lemma 4.5 *Let X, Y and Z be closed sets with $Y \subset Z$; let $h_1 \in C(X, Z)$ and $h_2 \in C(Y, X)$. Suppose $h_1 \circ h_2$ is homotopically equivalent to the identity mapping id in Z , then $\text{cat}_X(X) \geq \text{cat}_Z(Y)$.*

Proof of Theorem 1.2. We know that Φ_λ is C^1 and \mathcal{N}_λ^- is a $C^{1,1}$ complete Riemannian manifold. Also Φ_λ is bounded from below on \mathcal{N}_λ^- and satisfies (PS)-condition. By Theorem 4.1, Φ_λ has at least $\text{cat}_{\mathcal{N}_\lambda(d_\lambda)}(\mathcal{N}_\lambda(d_\lambda))$ critical points.

Let $G_\varepsilon(x) = u_\lambda^+ + t_z w_{\varepsilon, z} \in \mathcal{N}_\lambda^-$, where $t_z > 0$ depends on z (see the proof of Theorem 3.2). Then,

$$\Phi_\lambda(u_\lambda^+ + t_z w_{\varepsilon, z}) < d_\lambda = c_\lambda^+ + \frac{\alpha}{N} S_\alpha^{\frac{N}{2\alpha}}$$

for all $z \in M$ and $\lambda < \Lambda_0$, see Lemma 3.3. This implies $u_\lambda^+ + t_z w_{\varepsilon, z} \in \mathcal{N}_\lambda(d_\lambda)$. Therefore, by Lemma 4.4, we know $F \circ G_\varepsilon : M \rightarrow M_\delta$ is well defined. Next, we show that $F \circ G_\varepsilon$ is homotopically equivalent to the identity mapping on M_δ . In fact,

$$\begin{aligned} F(G_\varepsilon(z)) &= \frac{\int_\Omega x |G_\varepsilon(z)|^{2_\alpha^*} dx}{\int_\Omega |G_\varepsilon(z)|^{2_\alpha^*} dx} \\ &= \frac{\int_\Omega x |u_\lambda^+ + t_z w_{\varepsilon, z}|^{2_\alpha^*} dx}{\int_\Omega |u_\lambda^+ + t_z w_{\varepsilon, z}|^{2_\alpha^*} dx} \\ &= \frac{\int_\Omega (x+z) |u_\lambda^+(x+z) + t_z \eta(x) U_\varepsilon(x)|^{2_\alpha^*} dx}{\int_\Omega |u_\lambda^+(x+z) + t_z \eta(x) U_\varepsilon(x)|^{2_\alpha^*} dx} \\ &= z + \frac{\int_\Omega z |u_\lambda^+(x+z) + t_z \eta(x) U_\varepsilon(x)|^{2_\alpha^*} dx}{\int_\Omega |u_\lambda^+(x+z) + t_z \eta(x) U_\varepsilon(x)|^{2_\alpha^*} dx} \\ &= z + o(1) \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Applying Lemma 4.5 with $X = \mathcal{N}_\lambda(d_\lambda)$, $Y = M$, $Z = M_\delta$, $h_1 = F$ and $h_2 = G_\varepsilon$, we have $\text{cat}_{\mathcal{N}_\lambda(d_\lambda)}(\mathcal{N}_\lambda(d_\lambda)) \geq \text{cat}_{M_\delta}(M)$.

Finally, combining the above results with Theorem 3.1, we know problem (1.1) has at least $\text{cat}_{M_\delta}(M) + 1$ solutions. \square

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